Sunyaev Zel'dovich Effect

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Abstract

This report attempts to consolidate everything we've learnt about the Sunyaez Zel'dovich Effect (henceforth referred to as SZ Effect) beginning from the derivation of the Kompaneet's equation and culminating with the derivation of g̃(x) - a function simply related to the change in intensity of the Cosmic Microwave Background Radiation (CMBR) $\Delta I(x)$, from which the SZ Effect can be calculated.

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1 Derivation of Kompaneet's Equation

1.1 Compton Scattering

When an energetic photon interacts with a charged particle (say, an electron) it was found that, contrary to the classical view of light, the photon undergoes an inelastic with the particle. This leads to the reduction of the energy of the photon and provides the stationary or slow moving electron with a recoil velocity. This phenomenon is known as Compton Scattering and its discovery in 1923 was a further nail in the coffin of the classical theory which described light as a wave.

Now, Compton Scattering follows Energy Conservation and Momentum Conservation, thus from these two equations, we can arrive at an equation for the shift in the wavelength of the incident photon.

Let us now consider the scattering of a photon with a stationary electron. If the unprimed variables denote the electron (e) and photon (γ) before the scattering occurs and the primed variables denote the electron and the photon after the scattering, we can see that Energy Conservation and Momentum Conservation state that:

$$E_\gamma + E_e = E'_\gamma + E'_e \tag{1}$$

$$\overrightarrow{p_\gamma} = \overrightarrow{p'_\gamma} + \overrightarrow{p'_e} \tag{2}$$

Now, if h is Planck's Constant, c the speed of light, m_e the rest mass of the electron and ν and ν' the initial and final frequencies of the photon respectively, then:

$$E_\gamma = h\nu \qquad ; \qquad E'_\gamma = h\nu' \qquad ; \qquad E_e = m_e c^2$$

$$E'_e = \sqrt{(p'_e c)^2 + (m_e c^2)^2}$$

Substituting these relations in Eqn (1), we get:

$$h\nu + m_e c^2 = h\nu' + \sqrt{(p'_e c)^2 + (m_e c^2)^2} \tag{3}$$

which on rearranging and squaring would provide us with the equation:

$$p'^2_e c^2 = (h\nu + m_e c^2 - h\nu')^2 - m_e^2 c^4 \tag{4}$$

and on rearranging Eqn (2) we get:

$$\overrightarrow{p'_e} = \overrightarrow{p_\gamma} - \overrightarrow{p'_\gamma} \tag{5}$$

which when dotted with itself provides an expression for p'^2_e

$$\overrightarrow{p'_e}.\overrightarrow{p'_e} = (\overrightarrow{p_\gamma} - \overrightarrow{p'_\gamma}).(\overrightarrow{p_\gamma} - \overrightarrow{p'_\gamma}) = p_\gamma^2 + p'^2_\gamma - 2p_\gamma p'_\gamma \cos\theta \tag{6}$$

Thus $p'^2_e c^2$ is:

$$p'^2_e c^2 = (p_\gamma c)^2 + (p'_\gamma c)^2 - 2c^2 p_\gamma p'_\gamma \cos\theta \tag{7}$$

Now, using the relation $E = pc$, and equating the energies E_γ and E'_γ to the frequencies ν and ν' respectively, we get:

$$p'^2_e c^2 = (h\nu)^2 + (h\nu')^2 - 2(h\nu)(h\nu')\cos\theta \tag{8}$$

Substituting $p'^2_e c^2$ into Eqn (4):

$$(h\nu + m_e c^2 - h\nu')^2 - m_e^2 c^4 = (h\nu)^2 + (h\nu')^2 - 2(h\nu)(h\nu')\cos\theta \tag{9}$$

On expanding and rearranging the previous equation, we can see that:

$$2h\nu m_e c^2 - 2h\nu' m_e c^2 = 2h^2\nu\nu'(1 - \cos\theta) \tag{10}$$

Dividing the entire equation by $2h\nu\nu' m_e c$, it takes the form:

$$\frac{c}{\nu'} - \frac{c}{\nu} = \frac{h}{m_e c}(1 - \cos\theta) \tag{11}$$

Now using the relation $c/\nu = \lambda$, we can see that

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta) \tag{12}$$

Assigning $\Delta\lambda = \lambda' - \lambda$ and $\lambda_c = h/m_e c$ - called the 'Compton wavelength':

$$\Delta\lambda = 2\lambda_c(1 - \cos\theta) \tag{13}$$

1.2 Inverse Compton Scattering

One also encounters the opposite effect: of energetic particles transferring momentum to low energy photons. This process is called 'Inverse Compton Scattering'. These two apparently different phenomena can be thought of as the same process viewed from two different frames of reference. If the observer were at rest with respect to the high energy particle, the scattering would appear as Compton scattering, as the electron will appear to be at rest while the photon highly energetic.

Let us assume an electron that is not stationary and a photon of energy $h\nu$ colliding with it at an angle α to the electron's trajectory. Since the electron is moving at relativistic speeds, it will have an energy of $\gamma m_e c^2$ where γ is the Lorentz factor, given by: $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$

Now, shifting from our stationary frame to the rest frame of the electron, the incident photon's frequency will appear to be blue shifted, due to the relativistic doppler shift, and so the apparent frequency of the photon will be given by:

$$\nu' = \gamma\nu(1 - \frac{v}{c}\cos\theta) \tag{14}$$

Also, in the electron's rest frame, the angle α will transform to α' which will be given by the equations

$$\sin\alpha' = \frac{\sin\alpha}{\gamma[1 + (v/c)\cos\alpha]} \qquad \cos\alpha' = \frac{\cos\alpha + (v/c)}{1 + (v/c)\cos\alpha} \tag{15}$$

Now, the photon will appear to be blue shifted, with an energy:

$$h\nu' = \gamma h\nu(1 - \frac{v}{c}\cos\theta) \tag{16}$$

Now, these new equations for the new frequency (ν') and the new angle (α'), can be substituted into the earlier equations for Compton Scattering to provide us with the scattered wavelength and frequency (λ'' and ν'' in the electron's rest frame:

$$\lambda'' - \lambda' = \frac{h}{m_e c}(1 - \cos\alpha') \tag{17}$$

Substituting the relation $\lambda = c/\nu$ and the $\cos\alpha'$ from Eqn (15):

$$\frac{c}{\nu''} - \frac{c}{\nu'} = \frac{h}{m_e c}(1 - \cos\alpha') \tag{18}$$

Substituting ν' from Eqn (14) we get:

$$\frac{c}{\nu''} - \frac{c}{\gamma\nu[1 - (v/c)\cos\alpha]} = \frac{h}{m_e c}(1 - \frac{\cos\alpha + (v/c)}{1 + (v/c)\cos\alpha}) \tag{19}$$

Taking a few special cases:

a) For $\alpha = 0$ It isn't hard to show that the entire right hand side of the equation goes to zero. Thus:

$$\frac{c}{\nu''} = \frac{c}{\gamma\nu(1-(v/c))} \tag{20}$$

Using the approximation $1 - \frac{v}{c} = \frac{1-v^2/c^2}{1+(v/c)} \approx \frac{1}{2\gamma^2}$

We can see that $\nu'' = \frac{\gamma\nu}{2\gamma^2} = \frac{\nu}{2\gamma}$

b) Similarly, for $\alpha = \pi$ The velocity is now changed from v to $-v$ since the photon is now approaching from the other side. The equation now becomes:

$$\frac{c}{\nu''} - \frac{c}{\gamma\nu(1+(v/c))} = \frac{h}{m_e c}(1 - \frac{(v/c)-1}{1-(v/c)}) \tag{21}$$

On dividing the entire equation by c, multiplying by ν and applying the approximation $1 + (v/c) \approx 2$ (since the electron's velocity is comparable to the speed of light), we get:

$$\frac{\nu}{\nu''} - \frac{1}{2\gamma} = \frac{2h\nu}{m_e c^2} \tag{22}$$

Now, in the realm that we are interested in $h\nu \ll m_e c^2$ and so the entire right hand side of the equation goes to zero. On rearranging, we get:

$$\nu'' \approx 2\gamma\nu \tag{23}$$

c) For a more or less average case of $\alpha = \frac{\pi}{2}$

$$\frac{c}{\nu''} - \frac{c}{\gamma\nu} = \frac{h}{m_e c}(1 - (v/c)) \tag{24}$$

Dividing the equation by c and multiplying by ν we can use similar reasoning as the last time to show that the new frequency is now:

$$\nu'' \approx \gamma\nu \tag{25}$$

Now, all values of α are possible, and so for the 'average' value of $\alpha = \frac{\pi}{2}$ we see that

$h\nu'' \sim h\nu' \sim \gamma h\nu$

However, another transformation is necessary, to bring this back to the stationary frame, and thus the photon is again blue shifted by a further factor of γ due to the Lorentz Transformation. Thus, if ν_s is the frequency in the stationary frame, we have:

$$h\nu_s \sim \gamma h\nu'' \sim \gamma^2 h\nu \tag{26}$$

This result has important applications in astrophysics. One often encounters electrons with large values of γ (generally of the order 10^3), which scatter the incident low energy photon causing it to gain energy by a factor of γ^2 (10^6). Thus, any incident radio photons of frequency $\nu = 10^9 Hz$ will now be shifted up the spectrum to a $\nu' = 10^{15} Hz$

1.3 Derivation of expression for $\nu' - \nu$

The conservation of energy and momentum state that:

$$h\nu + \frac{\overrightarrow{p}^2}{2m_e} = h\nu' + \frac{\overrightarrow{p'}^2}{2m_e} \quad (27)$$

$$\frac{h\nu}{c}\hat{n} + \overrightarrow{p} = \frac{h\nu'}{c}\hat{n'} + \overrightarrow{p'} \quad (28)$$

where $\hat{n}$ and $\hat{n'}$ are the directions of the electron's initial and final trajectory. The unprimed values denote the values before the scattering and the primed denote values after the scattering.

Defining a new value $\Delta = \nu' - \nu$, we get from Eqn (27):

$$h\Delta = \frac{1}{2m_e}(\overrightarrow{p}^2 - \overrightarrow{p'}^2) \quad (29)$$

and on rearranging (28):

$$\overrightarrow{p'} = \overrightarrow{p} + \frac{h}{c}(\nu\hat{n} - \nu'\hat{n}') \quad (30)$$

Squaring the previous equation, we get:

$$p'^2 = \frac{h^2}{c^2}(\nu^2 + \nu'^2 - 2\nu\nu'\hat{n}\hat{n}') + p^2 + 2\frac{h\overrightarrow{p}}{c}.(\nu\hat{n} - \nu'\hat{n}') \quad (31)$$

Substituting $\nu' = \Delta + \nu$ we get:

$$p'^2 = \frac{h^2}{c^2}(\nu^2+\Delta^2+\nu^2+2\nu\Delta-2\nu(\Delta+\nu)\hat{n}\hat{n}')+p^2+2\frac{h\overrightarrow{p}}{c}.(\nu\hat{n}-(\Delta+\nu)\hat{n}') \quad (32)$$

Substituting this value of p'^2 into Eqn (29):

$$2m_eh\Delta = p^2-[\frac{h^2}{c^2}(2\nu^2+\Delta^2+2\nu\Delta(1-\hat{n}\hat{n}')-2\nu^2\hat{n}\hat{n}')+p^2+2\frac{h\overrightarrow{p}\nu}{c}(\hat{n}-\hat{n}')-2\frac{h\Delta\overrightarrow{p}.\hat{n}'}{c}] \quad (33)$$

Ignoring the terms of Δ^2, since we assume Δ to be very small, we get:

$$h\Delta(m_e + \frac{h\nu}{c^2}(1-\hat{n}\hat{n}') - \frac{\overrightarrow{p}.\hat{n}'}{c}) = -[\frac{h\nu\overrightarrow{p}.}{c}(\hat{n}-\hat{n}') + h^2\nu^2(1-\hat{n}\hat{n}')] \quad (34)$$

Multiplying the equation by c^2 and keeping $h\Delta$ to one side, we get the equation:

$$h\Delta = -\frac{hc\nu\overrightarrow{p}.(\hat{n}-\hat{n}') + h^2\nu^2(1-\hat{n}\hat{n}')}{(m_ec^2 + h\nu(1-\hat{n}\hat{n}') - c\overrightarrow{p}.\hat{n}'}) \quad (35)$$

Now, if we take $h\nu \sim k_BT \sim O(m_ev^2)$, where v is some characteristic electron thermal velocity, then we see that both the second term in the numerator and the third term in the denominator are $O(v/c)$ to their first terms respectively. The second term in the denominator is similarly an $O(v^2/c^2)$ to its first term, and thus taking only the leading terms in the previous equation, we have:

$$h\Delta \approx -\frac{h\nu\overrightarrow{p}.(\hat{n}-\hat{n}')}{m_ec} \quad (36)$$

1.4 Calculation of Boltzmann Equation for Photon Occupation Number

There are $n!$ ways of arranging n particles. If the particles are distinguishable, the probability of having n particles in a state is given by:

$$P_{dist}(n) = n!|u_a(x_1)u_a(x_2)...u_a(x_n|^2 \quad (37)$$

If, however, the particles are identical, the probability of having n such particles in a state is given instead by:

$$P_{indist}(n) = |n!u_a(x_1)u_a(x_2)...u_a(x_n|^2 = n!P_{dist}(n) \quad (38)$$

Now if p is the probability that a distinguishable particle goes into a state and the state already has n such particles in it, then the probability of adding another particle is:

$$P_{dist}(n+1) = pP_{dist}(n) \tag{39}$$

However, bosons are indistinguishable, and thus the probability of adding a boson to a state that already has n particles in it is:

$$P_{boson}(n+1) = (n+1)!P_{dist}(n+1) = (n+1)!pP_{dist}(n) = (n+1)n!pP_{dist}(n) \tag{40}$$

Thus, the probability of adding a boson (like a photon) to a state already containing n particles is enhanced by a factor of $(n+1)$. i.e:

$$P_{boson}(n+1) = (n+1)pP_{boson}(n) \tag{41}$$

We will use this fact to derive the Boltzmann distribution governing the evolution of the photon occupation number $\eta(\nu)$.

The rate of change of $\eta(\nu)$ with time can be represented as an integral over momentum space. Now, let us define the values $N(E)$ as the electron distribution function per unit momentum space and $d\sigma = (d\sigma/d\Omega)d\Omega$ as the infinitesimal element of the cross-section of scattering of a photon into the element of the solid angle $d\Omega$.

If we assume our electrons to be in thermal equilibrium, this will simply depend on the energy and not on the direction, since in thermal equilibrium no particular is favored. Also, the scattering cross-section σ of an interaction is the hypothetical area around the target particles that represents a surface within which there is some kind of interaction. Now, in time t, a photon would have covered a volume equal to $cd\sigma t$. The number of interactions in this volume would depend on the number of electrons present in it, i.e, the photon occupation number $\eta(\nu)$ and the electron distribution function $N(E)$. However, since photons are bosons, this will be enhanced by a factor $(1+\eta(\nu'))$ as shown before, as the state would already have $\eta(\nu')$ photons in it.

However, there will also exist photons of frequency ν' that are scattered to frequency ν and thus we would require a symmetric term relating $\eta(\nu')$, $(1+\eta(\nu))$ and $N(E')$ to be subtracted from the earlier term. This second term proceeds at a different rate only because the occupation numbers of photons and electrons are different. The scattering cross-section is the same for both, because if it were not so an equilibrium gas of photons and electrons would be able to drive itself away from equilibrium - a direct violation of the second law of thermodynamics.

The electrons are fermions, and thus a further terms corresponding to the fermion factor $(1-N(E'))$ and $(1-N(E))$ should technically be multiplied to each of the terms respectively. However, the electrons are assumed to be non-degenerate, which implies that $N \ll 1$ and so the factor is close to 1 and thus ignored.

By the earlier definition of $\Delta = \nu' - \nu$ we get:

$\nu' = \nu + \Delta$ and $E' = E - h\Delta$

Now, substituting the above values in an integral over momentum space we get:

$$\frac{\partial \eta(\nu)}{\partial t} = \int d^3p \;\, cd\sigma[\eta(\nu)(1+\eta(\nu'))N(E) - \eta(\nu')(1+\eta(\nu))N(E')] \tag{42}$$

We have already assumed the electron's distribution to be Maxwellian. Now, assuming that the photon distribution function is a general Bose distribution, then:

$$\eta(\nu) = \frac{1}{exp(a + h\nu/k_B T) - 1} \tag{43}$$

We can greatly simplify Eqn (42) by relating the primed coordinates to the unprimed ones in terms of a Taylor expansion in powers of Δ, which we assume to be very small:

In order to further simplify the equation, we will now define a new dimensionless constant $x \equiv h\nu/k_B T$. The expansion now simply becomes:

$$\eta(\nu') = \eta(\nu + \Delta) = \eta(\nu) + \frac{h\Delta}{k_B T}\frac{\partial \eta}{\partial x} + \frac{1}{2}(\frac{h\Delta}{k_B T})^2 \frac{\partial^2 \eta}{\partial x^2} + ... \tag{44}$$

and since $N(E)$ is an exponential function:

$$N(E') = N(E - h\Delta) = N(E)[1 + \frac{h\Delta}{k_B T} + \frac{1}{2}(\frac{h\Delta}{k_B T})^2 + ...] \tag{45}$$

We will now attempt to substitute Eqns (44) and (45) into Eqn (42). First let us define a new variable $\alpha \equiv h\Delta/k_B T$

$$\eta(\nu)(1 + \eta(\nu'))N(E) = \eta(1 + \eta + \alpha\frac{\partial n}{\partial x} + \frac{\alpha^2}{2}\frac{\partial^2 n}{\partial x^2}...)N(E) \tag{46}$$

and

$$\eta(\nu')(1 + \eta(\nu))N(E') = (\eta + \alpha\frac{\partial n}{\partial x} + \frac{\alpha^2}{2}\frac{\partial^2 n}{\partial x^2}...)(1 + \eta(\nu))N(E)(1 + \alpha + \frac{\alpha^2}{2}...) \tag{47}$$

Ignoring all terms greater that α^2 and expanding the last equation we get:

$$\eta(\nu')(1+\eta(\nu))N(E') = [\eta + \alpha\frac{\partial n}{\partial x} + \frac{\alpha^2}{2}\frac{\partial^2 n}{\partial x^2} + \eta\alpha + \alpha^2\frac{\partial n}{\partial x} + \frac{\alpha^3}{2}\frac{\partial^2 n}{\partial x^2} + \frac{\eta\alpha^2}{2} + \frac{\alpha^3}{2}\frac{\partial n}{\partial x} + ... \quad (48)$$

$$... \frac{\alpha^4}{4}\frac{\partial^2 n}{\partial x^2} + \eta^2 + \alpha\eta\frac{\partial n}{\partial x} + \frac{\alpha^2\eta}{2}\frac{\partial^2 n}{\partial x^2} + \alpha\eta^2 + \alpha^2\eta\frac{\partial n}{\partial x} + \frac{\alpha^3\eta}{2}\frac{\partial^2 n}{\partial x^2} + ... \quad (49)$$

$$... \frac{\alpha^2\eta^2}{2} + \frac{\alpha^3\eta}{2}\frac{\partial n}{\partial x} + \frac{\alpha^4\eta}{4}\frac{\partial^2 n}{\partial x^2}]N(E) \quad (50)$$

On subtracting these equations and collecting powers of α we get the difference equal to:

$$N(E)[\alpha(\frac{\partial \eta}{\partial x} + \eta + \eta^2) + \frac{\alpha^2}{2}(\frac{\partial^2 \eta}{\partial x^2} + 2\frac{\partial \eta}{\partial x} + \eta + 2\eta\frac{\partial \eta}{\partial x} + \eta^2)]$$

Now substituting this difference into the integral given in Eqn (42) and also removing α we get:

$$\frac{\partial \eta(\nu)}{\partial t} = \int [\frac{h\Delta}{k_B T}(\frac{\partial \eta}{\partial x} + \eta + \eta^2) + \frac{1}{2}(\frac{h\Delta}{k_B T})^2(\frac{\partial^2 \eta}{\partial x^2} + 2(1+\eta)\frac{\partial \eta}{\partial x} + \eta + \eta^2)]N(E)c \; d\sigma \, d^3p \quad (51)$$

Let us define two new integrals I_1 and I_2 that are:

$$I_1 \equiv \int d^3p \;\; c d\sigma N(E)\Delta$$

$$I_2 \equiv \int d^3p \;\; c d\sigma N(E)\Delta^2$$

The earlier equation now reduces to:

$$\frac{\partial \eta(\nu)}{\partial t} = \frac{h}{k_B T}(\frac{\partial \eta}{\partial x} + \eta + \eta^2)I_1 + \frac{1}{2}(\frac{h}{k_B T})^2(\frac{\partial^2 \eta}{\partial x^2} + 2(1+\eta)\frac{\partial \eta}{\partial x} + \eta + \eta^2)I_2 \quad (52)$$

with us only needing now to calculate the values of I_1 and I_2. With any luck, we will not need to find the value of I_2 and use reasoning to find the form of I_1.

Let us now substitute the value of Δ Eqn (36) into the expression of I_2 to get:

$$I_2 = (\frac{\nu}{m_e c})^2 \int c \; d\sigma \; d^3p \; N(E) \; (\vec{p}.(\hat{n} - \hat{n}'))^2 \tag{53}$$

Let us now write $\vec{p}.(\hat{n} - \hat{n}') = p.|\hat{n} - \hat{n}'| \cos\psi$ where ψ is the included angle between the two vectors. The quantity $|\hat{n} - \hat{n}'|^2$ does not depend on $\vec{p}$ and so can be taken out of the integral over electron momentum space.

The integral now has an angular integral of $\cos^2\psi$ which is $4\pi/3$ for any polar axis. Performing the integral over the direction of $\vec{p}$

$$I_2 = \frac{1}{3}(\frac{\nu}{m_e c})^2 \int c \; d\sigma \; |\hat{n} - \hat{n}'|^2 \int 4\pi p^2 dp \; N(E) \, p^2 \tag{54}$$

The last integral is now merely n_e times $\langle p^2 \rangle$, or $2m_e$ times the electron kinetic energy density. Since we have assumed $N(E)$ to be Maxwellian, this has a value of $3k_B T m_e n_e$ Then

$$I_2 = (\frac{\nu}{m_e c})^2 k_B T m_e n_e c \int d\Omega \frac{d\sigma}{d\Omega} \; |\hat{n} - \hat{n}'|^2 \tag{55}$$

The exact differential cross-section for Compton Scattering is given by the Klein-Nishina formula. However, since we are working in the non-relativistic limit, we approximate it by the Thomson differential cross-section

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} r_e^2 (1 + (\hat{n}.\hat{n}')^2)$$
$$r_e \equiv \frac{e^2}{m_e c^2}$$

where r_e is the classical electron radius. This would include a fractional error of $O(v^2/c^2)$, but since we are considering the leading terms in expansions whose successive terms are $O(v/c)$, this error can be ignored. Similarly, we ignore the Lorentz Transformations between the stationary and scattering frames in evaluating the differential cross-sections as in the non-relativistic limit these frames are identical. A complete relativistic treatment would include this, but would be too cumbersome.

Substituting this value for the differential cross section and also substituting $|\hat{n}-\hat{n}'|^2 = \hat{n}^2 + \hat{n}'^2 - 2\hat{n}\hat{n}' = 2 - 2(\hat{n}\hat{n}')$

$$I_2 = (\frac{\nu}{m_e c})^2 k_B T m_e n_e c \int d\Omega \frac{1}{2} r_e^2 (1 + (\hat{n}.\hat{n}')^2)\ (2 - 2(\hat{n}\hat{n}')) \tag{56}$$

This integral is now carried over the photon scattering angle $d\Omega$. All integrals over spheres of odd powers of $\hat{n}\hat{n}' = \cos\theta$ are zero, while the integral of $\cos^2\theta$ is (as shown earlier) $4\pi/3$. If we write the integral now in terms of the Thompson cross-section $\sigma_{es} = \frac{8}{3}\pi r_e^2$, we get the value of the integral to be:

$$I_2 = 2(\frac{\nu}{m_e c})^2 k_B T m_e n_e \sigma_{es} c \tag{57}$$

Substituting this equation into we see that the integral of Δ^2 contributes a term proportional to $x^2(\partial^2\eta/\partial x^2)$ to the $\partial\eta/\partial t$ term. This result will permit the derivation of the Kompaneet's Equation without further computation of integrals.

We know that Compton Scattering conserves photons, and this the photon occupation number $\eta(\nu)$ must satisfy a conservation law in three dimensional momentum space. Since we have a isotropic photon distribution, η can depend only on the magnitude of the photon momentum and not the angle or distance and thus the law takes the form:

$$\frac{\partial\eta}{\partial t} = -\frac{1}{x^2}\frac{\partial(x^2 j)}{\partial x} \tag{58}$$

where j is the 'current density' of the photons. Let us now observe Eqn (48), which is:

$$\frac{\partial\eta(\nu)}{\partial t} = \int[\frac{h\Delta}{k_BT}(\frac{\partial\eta}{\partial x}+\eta+\eta^2)+\frac{1}{2}(\frac{h\Delta}{k_BT})^2(\frac{\partial^2\eta}{\partial x^2}+2(1+\eta)\frac{\partial\eta}{\partial x}+\eta+\eta^2)]N(E)c\ d\sigma\, d^3p$$

As can be seen, this equation contains a term equal to $\partial^2\eta/\partial x^2$ times a function of x but not of η. This greatly allows us to simplify our final equation. Now, Eqn (58) describes the same function, and thus must have the same form. On expanding it, we get:

$$\frac{\partial\eta}{\partial x} = -\frac{1}{x^2}\frac{\partial(x^2 j)}{\partial x} = -\frac{2}{x}j - \frac{\partial j}{\partial x} \tag{59}$$

Clearly from this we can see that since the coefficient of $\partial^2\eta/\partial x^2$ is not dependant on η, any term in j that is proportional to $\partial\eta/\partial x$ must depend only on x. Therefore, we can now provide a general form for $j(\eta, x)$:

$$j(\eta, x) = k(x)(\frac{\partial\eta}{\partial x} + h(\eta, x)) \tag{60}$$

Now if we manage to determine the functions k and h, we will have the complete form of $\partial\eta/\partial x$. To do this, we first observe that we have assumed the photon distribution to be general Bose distribution, and the number of photons is conserved, thus there are no photon sources or sinks. From this, we can clearly see that $j = 0$ for this 'stationary solution'.

Using the general Bose distribution earlier provided, $\eta = \frac{1}{e^x-1}$ and thus we can clearly see that:

$$\frac{\partial\eta}{\partial x} = -\frac{e^x}{(e^x-1)^2} = -\eta - \eta^2 \tag{61}$$

On substituting this and the condition that $j = 0$ into our new equation for j and rearranging, we arrive at:

$$h(\eta, x) = \eta + \eta^2 \tag{62}$$

for all η and x.

Now to determine $k(x)$ we compare the coefficients of $\partial^2\eta/\partial x$ from the earlier two equations, we see that the coefficient is proportional to ν^2. Thus, $k(x) \propto \nu^2 \propto x^2$. Comparing the constants, we see that:

$$k(x) = -\frac{k_B T}{m_e c^2} n_e \sigma_{es} c x^2 \tag{63}$$

Let us now define a new dimensionless scaled quantity y that will make our equation neater.

$$y \equiv t \frac{k_B T}{m_e c^2} n_e \sigma_{es} c \tag{64}$$

Substituting the last three equations into our equation, we get Kompaneets Equation:

$$\frac{\partial \eta}{\partial y} = \frac{1}{x^2} \frac{\partial}{\partial x} [x^4 \ (\frac{\partial \eta}{\partial x} + \eta + \eta^2)] \tag{65}$$

1.5 Kompaneets Equation and Comptonization: The Sunyaev Zel'dovich Effect

In general, when high energy photons and electrons interact, the exchange of energy between them leads to distinct changes in the photons' spectrum and the energy content of the electrons. If this interaction occurs due to Compton scattering, the term Comptonization is used. In the case of - say - non-relativistic, highly thermal electrons and photons with a Planckian spectrum, this transfer will eventually lead to the balancing out of both the spectrums.

However, there exist some astronomical situations in which the electron temperature is much higher than the temperature of the blackbody radiation. In such cases, the spectrum of the photons is changed significantly from a Planckian spectrum. In a non-relativistic case, the evolution of this spectrum is described by the Kompaneet's equation that was derived in the last section. The parameter y that we described in the last section may be thought of as a degree of interaction between the photons and electrons. y can be described as:

$$y = \frac{k_B T_e}{m_e c^2} n_e \sigma_{es} ct = \frac{k_B T_e}{m_e c^2} n_e \sigma_T L \tag{66}$$

Where T_e is the temperature of the electrons, n_e their number density, σ_{es} the scattering cross-section (which is the Thompson cross-section σ_T for the non-relativistic limit) and $ct = L$ the size of the gaseous system.

We will now try to simplify this equation. A few assumptions will allow us to do this. For a start, let us assume $T_e \gg T_\gamma$ where T_γ is the temperature of the photons. This is justified in almost all the cases where Kompaneets Equation is of any interest to us. Let us also assume that the distortion of the spectrum is small, which will allow us to write:

$$\eta \sim \frac{1}{e^{fx} - 1} \tag{67}$$

where f is some distortion factor, and can be given by $f = T_\gamma / T_e$

Using this η, we can calculate see that

$$\frac{\partial \eta}{\partial x} = \frac{-fe^{fx}}{(e^{fx} - 1)^2}$$
$$\frac{\partial^2 \eta}{\partial x^2} = \frac{\partial}{\partial x}(-fe^{fx}\eta^2) = -[f^2\eta^2 e^{fx} + 2f\eta e^{fx}\frac{\partial \eta}{\partial x}]$$

From here, calculating the value of $\eta^2 + \eta + \partial\eta/\partial x$ is a small step away:

$$\eta^2+\eta+\frac{\partial\eta}{\partial x} = \frac{1}{(e^{fx}-1)^2}+\frac{1}{(e^{fx}-1}-\frac{fe^{fx}}{(e^{fx}-1)^2} = \frac{(1-f)e^{fx}}{(e^{fx}-1)^2} = (1-f^{-1})\frac{\partial\eta}{\partial x} \tag{68}$$

Substituting this into Kompaneets Equation, we get:

$$\frac{\partial\eta}{\partial y} = \frac{(1-f^{-1})}{x^2}\frac{\partial}{\partial x}(x^4\frac{\partial\eta}{\partial x}) = (1-f^{-1})[x^2\frac{\partial^2\eta}{\partial x^2}+4x\frac{\partial\eta}{\partial x}] \tag{69}$$

Substituting the earlier values into this equation, (and doing some sloppy math) we get:

$$\frac{\Delta\eta}{\Delta y} = (1-f^{-1})[\frac{-4xfe^{fx}}{(e^{fx})-1)^2}-\frac{x^2f^2e^{fx}}{(e^{fx})-1)^2}-\frac{2f^2x^2e^{2fx}}{(e^{fx})-1)^3}]$$
$$\frac{\Delta\eta}{\eta} = \Delta y(1-f^{-1})[\frac{-4xfe^{fx}}{(e^{fx})-1)}-\frac{x^2f^2e^{fx}}{(e^{fx})-1)}-\frac{2f^2x^2e^{2fx}}{(e^{fx})-1)^2}]$$

Now, our $x = h\nu/k_BT_e$ as earlier defined. If we redefine $x = h\nu/k_BT_\gamma$, i.e, change $fx \to x$, the equation can be simplified. Also, our distortion factor $f \gg 1$, as we have assumed $T_e \gg T_\gamma$ and so $1 - \frac{1}{f} \approx 1$. Thus our equation now becomes:

$$\frac{\Delta\eta}{\eta} = \Delta y[-\frac{4xe^x}{e^x-1}-\frac{x^2e^x(e^x-1)-2x^2e^{2x}}{(e^x-1)^2}]$$
$$\frac{\Delta\eta}{\eta} = \Delta y[-\frac{4xe^x}{e^x-1}-\frac{x^2e^x(-e^x+1+2e^x)}{(e^x-1)^2}]$$

The ratio can now be written simply as:

$$\frac{\Delta\eta}{\eta} = \Delta y[\frac{x^2e^x(e^x+1)}{(e^x-1)^2}-\frac{4xe^x}{e^x-1}] \tag{70}$$

Now, this is in the Rayleigh-Jeans part of the spectrum, in the limit $x \ll 1$, this ratio approaches $-2\Delta y$ as can be seen, using the approximation $\lim_{x\to 0} \frac{e^x-1}{x} = 1$. Now, in the Rayleigh-Jeans region of the spectrum, $\eta \propto T_\gamma$ and thus we see that the temperature of the blackbody photons suffers a decrement of the order:

$$\frac{\Delta T_\gamma}{T_\gamma} \sim -2\Delta y \tag{71}$$

Let y_{th} be used to refer to Δy, the 'Comptonisation parameter'. This result may now be applied to photons which are of the microwave background radiation of the universe that pass through large reservoirs of hot gas present in galactic clusters and are then detected by radio telescopes. In typical clusters, $n_e \sim 10^{-3} cm^{-3}$ and a temperature $T_e \sim 5 \times 10^7 K$. The average size of such clusters is of the order $6 \times 10^{23} cm$. The distortion of the Rayleigh-Jeans part of the spectrum can be calculated to be:

$$\frac{\Delta T_\gamma}{T_\gamma} \sim 10^{-5} \tag{72}$$

Although this is a small change, it has been recently detected. First predicted in Sunyaev and Zel'dovich (1969) this is known as the Sunyaev and Zel'dovich (SZ) Effect. This is now increasingly becoming an important tool to detect and analyse regions of hot gas in the universe. This effect has the added advantage of providing us with an absorption spectrum instead of an emission spectrum. An emission spectrum would decrease in intensity with increase in red-shift and thus it'd be harder to detect objects further away. However, an absorption spectrum will retain its clarity independent of distance. Thus we now have a way to detect galactic clusters independent of red-shift.

2 Comptonisation of the CMB

There are many processes by which the CMB can be Comptonised, but we will restrict ourselves to the case of a non-thermal distribution of electrons and if possible, a thermal distribution as well. However, before we begin describing the non-thermal SZ effect, it would be better to adopt a generalised approach.

2.1 The SZ Effect for galactic clusters: A generalised approach

In this section we will attempt to derive a general expression for the SZ effect that is valid in the Thomson limit for a general electron population in the relativistic limit. We will also attempt to include the effects of multiple scattering.

We define the electron's normalised momentum as $p = \beta\gamma$, where as shown earlier $\gamma = 1/\sqrt{1 + v^2/c^2}$ and $\beta = v/c$. We arrive at this by dividing the electron's relativistic momentum γmv, (where v is its velocity) by a factor of mc.

Now, an electron with such a momentum increases the frequency ν of the CMB photon by scattering it to a new frequency ν'. Thus, it is convenient to define a new quantity that represents this factor. Thus, we define $t \equiv \nu'/\nu$. As already shown in the previous section while deriving Kompaneet's Equation, this factor $t \propto \gamma^2$, and so photons are scattered to much higher frequencies.

Now, since we're interested in large frequency shifts ($t \gg 1$) it is convenient to use a logarithmic scale for frequency shifts. We thus define an $s \equiv ln(t)$. The CMB photons' spectrum will be redistributed by the electron population, and this redistribution function writes as:

$$P_1(s) = \int_0^\infty dp f_e(p) P_s(s;p) \tag{73}$$

Where $f_e(p)$ is the electron momentum distribution and $P_s(s;p)$ is the redistribution function for a mono-energetic electron function. Once we determine $P_1(s)$, it is possible to evaluate the probability that a frequency change of s is produced by a number n of repeated, multiple scatterings. Thus, a repeated convolution is used to determine $P_n(s)$. i.e:

$$P_n(s) = \underbrace{P_1(s) \otimes .... \otimes P_1(s)}_{ntimes} \tag{74}$$

where $\otimes$ represents each convolution product.

Now, the resulting total redistribution function P_s is written as a sum of all the functions $P_n(s)$, with each of the terms weighted by the probability that a CMB photon suffers n scatterings. We will assume that the there are a large number of n's and the probability of a photon being scattered to any one is very small. Clearly, this distribution will be Poissonian, as we will show.

2.2 Derivation of a Poissonian Distribution

Let us assume we are tossing a biased coin N times, and since the coin in biased, the probability of getting a head is very low, i.e, $p \ll 1$

The probability $P(N,n)$ of getting n heads in N tosses is given by the binomial distribution, which can be written as:

$$P(N,n) = {}^N\mathbb{C}_n p^n (1-p)^{N-n} = \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n} \tag{75}$$

Now, we can make a few approximations, as we know

$ln(1-p) \approx -p$, if $p \ll 1$.

Thus, $ln[(1-p)^{N-n}] = (N-n)ln(1-p) \approx -Np + np$

Since our $n \ll N$, the coin being biased, we have:

$$(1-p)^{N-n} \approx e^{-Np}$$

Also, $ln(N!/(N-n)!) \approx NlnN - N - (N-n)ln(N-n) + (N-n) \approx nlnN$

Thus, we have:

$$\frac{N!}{(N-n)!} \approx N^n$$

Substituting all this into the last equation and substituting $\lambda = Np$ we get:

$$P(N,n) \approx \frac{\lambda^n e^{-\lambda}}{n!} \tag{76}$$

which is the Poissonian Distribution.

Thus, we use an expected value τ - which we will later discover is the optical depth - we write $P(s)$ as:

$$P(s) = \sum_{n=0}^{+\infty} \frac{e^{-\tau}\tau^n}{n!} P_n(s) = e^{-\tau}[P_0(s) + \tau P_1(s) + \frac{1}{2}\tau^2 P_2(s) + ...] \tag{77}$$

$$P(s) = e^{-\tau}[\delta(s) + \tau P_1(s) + \frac{1}{2}\tau^2 P_1(s) \otimes P_1(s) + ...] \tag{78}$$

Now, the spectrum of the incident CMB spectrum in terms of the a-dimensional x that we defined while deriving Kompaneets Equation, is:

$$I_0(x) = 2\frac{(k_B T_0)^3}{(hc)^2}\frac{x^3}{e^x - 1} \tag{79}$$

which we arrive at from Planck's law, since we assume the CMB spectrum to be Planckian.

The spectrum of the Comptonised radiation is given by:

$$I(x) = \int_{-\infty}^{+\infty} I_0(xe^{-s})P(s)ds \tag{80}$$

Since $x = h\nu'/k_BT_e$, the term xe^{-s} refers to a value of $x = h\nu/k_BT_e$, i.e the incident frequency.

2.3 High order τ expansion

In the calculation of the SZ effect in clusters, it is helpful to use an expression of $P(s)$ in terms of powers of τ. To do this, we will make use of a general expression for $P(s)$ in terms of a series expansion.

$$P(s) = \sum_{n=0}^{+\infty} a_n(s)\tau^n \tag{81}$$

Using Eqn (77) and substituting a series expansion for $e^{-\tau}$, we can see that this becomes now:

$$P(s) = \sum_{k=0}^{+\infty} \frac{(-\tau)^k}{k!} \sum_{k'=0}^{+\infty} \frac{(\tau)^{k'}}{k'!} P_{k'}(s) \tag{82}$$

Now, the general nth order term is obtained by selecting the terms in the summation that contain the optical depth τ up to the nth power. These terms are obtained for $k' = n - k$ and thus we see that:

$$P(s) = \sum_{k=0}^{n} \frac{(-\tau)^k}{k!} \frac{(\tau)^{(n-k)}}{(n-k)!} P_{n-k}(s) = \sum_{k=0}^{n} \frac{(-1)^k(\tau)^n}{k!(n-k)!} P_{n-k}(s) \tag{83}$$

Comparing this with Eqn (81) we get

$$a_n(s) = \frac{(-1)^k}{k!(n-k)!} P_{n-k}(s) \tag{84}$$

Substituting this in Eqn (81) and Eqn (81) in Eqn (80), we see that the distorted spectrum can itself be written as a series expansion:

$$I(x) = \sum_{n=0}^{+\infty} b_n(s)\tau^n \tag{85}$$

which can be given by:

$$I(x) = \frac{(-1)^k}{k!(n-k)!}\int_{-\infty}^{+\infty} I_0(xe^{-s})P_{n-k}(s)ds \tag{86}$$

Defining a new function $J_{n-k}(x)$, we have:

$$J_{n-k}(x) = \int_{-\infty}^{+\infty} I_0(xe^{-s})P_{n-k}(s)ds \tag{87}$$

Thus, we can see that:

$$J_0(x) = \int_{-\infty}^{+\infty} I_0(xe^{-s})P_0(s)ds = \int_{-\infty}^{+\infty} I_0(xe^{-s})\delta(s)ds = I_0(x) \tag{88}$$

and

$$J_1(x) = \int_{-\infty}^{+\infty} I_0(xe^{-s})P_1(s)ds \tag{89}$$

and so on.

Now, taking Eqn (78) and substituting it into Eqn (80) we get:

$$I(x) = \int_{-\infty}^{+\infty} I_0(xe^{-s})e^{-\tau}[\delta(s) + \tau P_1(s) + \frac{1}{2}\tau^2 P_1(s) \otimes P_1(s) + ...]ds \tag{90}$$

For small values of τ, $e^{-\tau} \approx (1-\tau)$. Using this and ignoring terms of higher powers of τ, we see:

$$I(x) = \int_{-\infty}^{+\infty} I_0(xe^{-s})(1-\tau)[\delta(s) + \tau P_1(s)]ds \tag{91}$$

$$I(x) = \int_{-\infty}^{+\infty} I_0(xe^{-s})\delta(s)ds + \tau \int_{-\infty}^{+\infty} I_0(xe^{-s})(P_1(s) - \delta(s)) \tag{92}$$

Using Eqns (88) and (89) we can see that this is very obviously:

$$I(x) = J_0(x) + \tau[J_1(x) - J_0(x)] \tag{93}$$

If necessary, higher powers of τ can be calculated, by expanding our $e^{-\tau}$ to a higher degree.

It is useful to write the distorted spectrum in the form:

$$\Delta I(x) = 2\frac{(k_B T_\gamma)^3}{(hc)^2} y\tilde{g}(x) \tag{94}$$

Where $\Delta I(x) \equiv I(x) - I_0(x)$ and y is the Comptonisation parameter and $\tilde{g}(x)$ is some function of x. If we determine $\tilde{g}(x)$ we can determine ΔI

2.4 General derivation of $\tilde{g}(x)$

From the previous expression we see that if we substitute the Planckian spectrum $I_0(x)$ of Eqn (79):

$$\tilde{g}(x) = \left(\frac{\Delta I}{I_0}\right) \frac{1}{y} \frac{x^3}{e^x - 1} = \frac{\Delta i(x)}{y} \tag{95}$$

Where $\Delta i(x) \equiv \Delta I \frac{(hc)^2}{2(K_B T_\gamma)^3}$ and y the Comptonisation parameter can be obtained in general, in terms of the pressure P of the considered electron population:

$$y = \frac{\sigma_T}{m_e c^2} \int P dl \tag{96}$$

From Eqn (88) we see that $J_0(x) = I_0(x)$ and so Eqn (93) now becomes:

$$\Delta I(x) = \tau[J_1(x) - J_0(x)] \tag{97}$$

Defining a new function $j_i(x) = J_i(x)\frac{(hc)^2}{2(K_BT_0)^3}$, Eqn (95) can not be written as:

$$\tilde{g}(x) = \frac{\tau[j_1(x) - j_0(x)]}{y} \tag{98}$$

This is of course, only up to first order terms of τ. As stated earlier, by limiting the series expansion at a higher order of τ we can get more accurate values of $\tilde{g}(x)$. Thus, limiting it at the third order in τ we see:

$$\tilde{g}(x) = \frac{1}{y}\left[\tau[j_1 - j_0] + \frac{1}{2}\tau^2(j_2 - 2j_1 + j_0) + \frac{1}{6}\tau^3(j_3 - 3j_2 + 3j_1 - j_0)\right] \tag{99}$$

y depends on the respective case (i.e, if the situation is thermal or non-thermal etc.), and we will see that it depends on τ. Thus, it is interesting to see that the first order approximation of $\tilde{g}(x)$ in τ is independent of τ.

From this it is obvious that to calculate $\tilde{g}(x)$ we need to calculate $j_n(x)$ for which we need to calculate $J_n(x)$ for which we need $P_n(x)$, which as we've already seen is a convolution of $P_1(x)$, n times. Thus, we must determine $P_1(x)$.

Using Eqn (73) we can obtain $P_1(s)$ if we know the values of $f_e(p)$ and $P_s(s;p)$. Now, $f_e(p)$ depends again on whether the situation is thermal or non-thermal, etc. However, an expression for $P_t(t;p)$ is given by Ensslin & Kaiser (2000):

$$P_t(t;p) = -\frac{3|1-t|}{32p^6t}[1+(10+8p^2+4p^4)t+t^2]+\frac{3(1+t)}{8p^5}[\frac{3+3p^2+p^4}{\sqrt{1+p^2}}-\frac{3+2p^2}{2p}(2asinh(p)-|ln(t)|)] \tag{100}$$

From the paper we see that the maximal frequency shift is given by:

$$|ln(t)| \leq 2asinh(p) \tag{101}$$

Thus, there can be no frequency shift greater than this and therefore $P_t(t;p) = 0$ for $|ln(t)| > 2asinh(p)$. Now, this is related to the $P_s(s;p)$ that we require simply by:

$$P_s(s;p)ds = P_s(e^s;p)\, e^s ds = P_t(t;p)\, t ds \tag{102}$$

Which can be computed numerically. From this, by substituting a value for $f_e(p)$ and integrating over all p's from some p_1 to p_2, we can arrive at an expression for $P_1(s)$ from which $P_n(s)$ can be calculated through repeated convolution, which after another numerical integral and substitution provides us with $j_n(x)$ which can be substituted in the equation for $\tilde{g}(x)$ to obtain it to desired accuracy.

2.5 The SZ effect for a non-thermal electron population

Now, using this general approach that we have derived, we proceed to derive the exact features of the spectral change produced by a single non-thermal population of electrons. Let us assume a single power-law electron population, which is described by the momentum spectrum:

$$f_e(p;p_1,p_2,\alpha) = A(p_1,p_2,\alpha)p^{-\alpha} \qquad p_1 \leq p \leq p_2 \tag{103}$$

Where $A(p_1,p_2,\alpha)$ is the normalisation term, given by:

$$A(p_1,p_2,\alpha) = \frac{(\alpha-1)}{p_1^{1-\alpha} - p_2^{1-\alpha}} \tag{104}$$

During the calculation of the non-thermal case, we will consider the minimum momentum p_1 of the electron distribution as a free parameter since it is not constrained by observational evidence, and the specific value of p_2 is irrelevant for power-law indices $\alpha > 2$.

Now,

$$\Delta I_{non-th}(x) = 2\frac{(k_B T_\gamma)^3}{(hc)^2} y_{non-th}\tilde{g}(x) \tag{105}$$

The Comptonisation parameter y is obtained by the expression:

$$y_{non-th} = \frac{\sigma_T}{m_e c^2} \int P_{rel}\, dl \tag{106}$$

Now, the energy per unit volume - i.e, P_{rel} of the electron distribution can be obtained by multiplying the energy of each electron with the momentum distribution $f_e(p)$ and integrating over all p' s from p_1 to p_2. The energies of each electron are obtained by the formula $E = \sqrt{(pc)^2 + (m_e c^2)^2}$.

Thus:

$$P_{rel} = \frac{E}{V} = \int_{p_1}^{p_2} \sqrt{(pc)^2 + (m_e c^2)^2} f_e(p) dp \tag{107}$$

This integral can be calculated to find a value that is sort of analogous to the 'equivalent temperature' $\langle K_B T_e \rangle$ for a thermal population of electrons.

Now, y_{non-th} becomes:

$$y_{non-th} = \frac{\sigma_T}{m_e c^2} \int P_{rel}\, dl = \sigma_T \frac{\langle K_B T_e \rangle}{m_e c^2} \int n_e\, dl = \frac{\langle k_B T_e \rangle}{m_e c^2} \tau \tag{108}$$

since $\tau = \sigma_T \int n_e\, dl$, by definition. Note that in this case, $\tau = \tau_{rel}$.

Substituting this into Eqn (105) we get:

$$\tilde{g}(x) = \frac{\Delta i}{y_{non-th}} = \frac{\tau [j_1 - j_0]}{\tau \frac{\langle k_B T_e \rangle}{m_e c^2}} \equiv \frac{m_e c^2}{\langle k_B T_e \rangle} [j_1 - j_0] \tag{109}$$

Thus, calculating $\langle k_B T_e \rangle$ for our electron population, we substitute it into the previous equation to get the value of $\tilde{g}(x)$ which we then substitute into Eqn (105) to get the function $\Delta I_{non-th}(x)$. It is interesting to note that in Eqn (105), the orders of all the terms except y_{non-th} cancel out, and thus the order of $\Delta I_{non-th}(x)$ is the same as that of y_{non-th}, which is around 10^{-5}, as was expected from Kompaneets Equation.

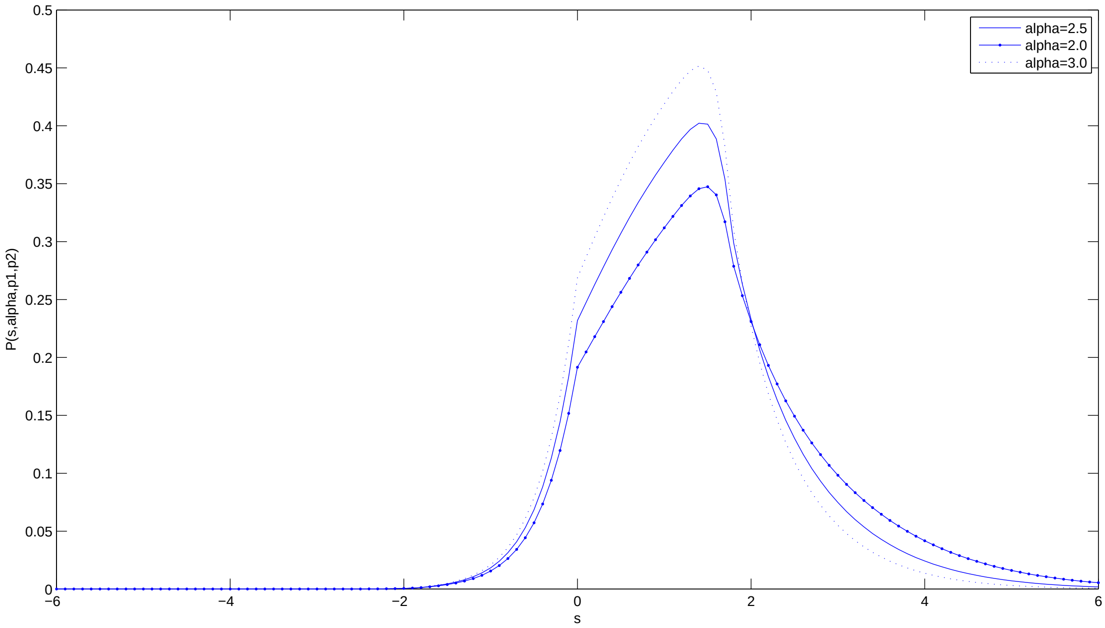

Figure 1: The CR-frequency redistribution function P(s;α,p1,p2)over s, with p1=1 and p2=10

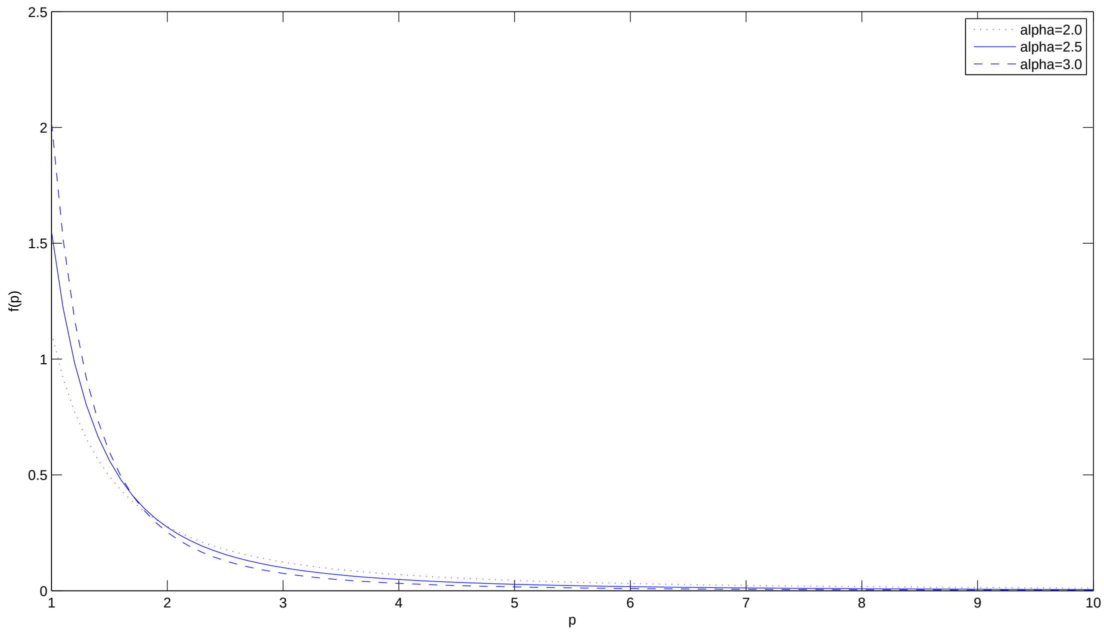

Figure 2: Momentum spectrum of electron population following single power-law, with $\alpha = 2$, 2.5 and 3

As seen from Fig(3) the zero of the non-thermal SZ effect produced by a single non-thermal population is shifted to a higher frequency as the index α is decreased. This can be explained from the fact that the number density of electrons with higher momentum is more for lower index (α) power law spectrum (Fig.(2)). Hence, the distortion of CMB spectrum is more from electron population having lower α. The zero of the non-thermal SZ effect produced by a single non-thermal population is also shifted to frequencies much higher than the value $x_{0,th} = 3.83$, due to large frequency shifts experienced b the CMB photons scattering the high energy non-thermal electrons

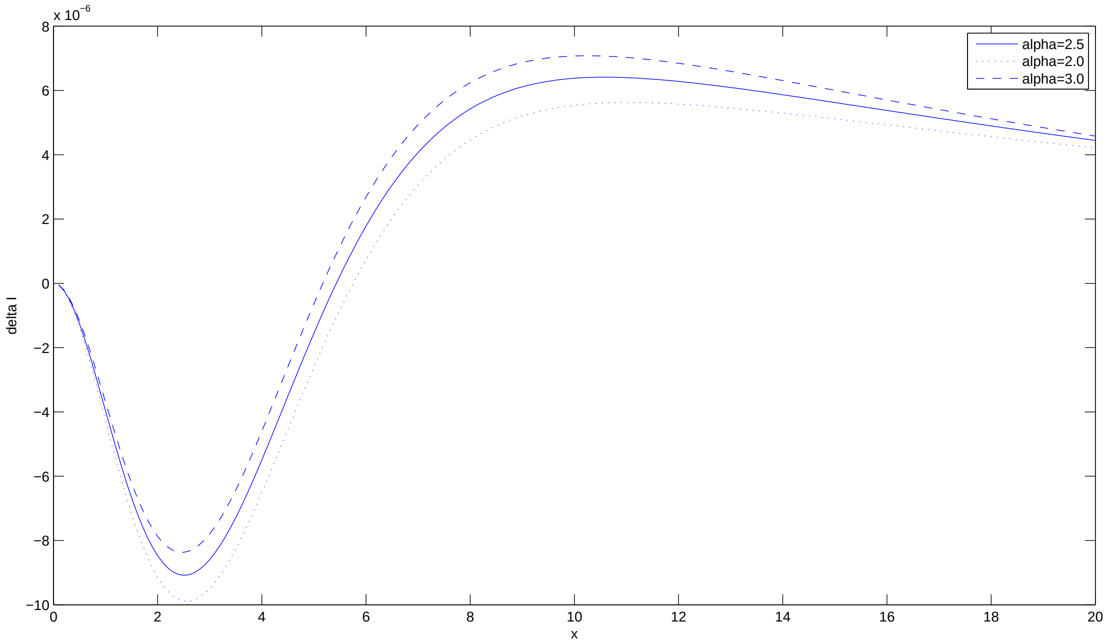

Figure 3: Change in Intensity produced due to electron population following single power law, with α =2, 2.5 and 3

2.6 The SZ Effect for a mono-energetic electron population

Again, using the general approach, we proceed to derive the exact features of the spectral change produced by mono-energetic population of electrons. Let us assume mono-energetic electron population, which is described by the momentum spectrum:

$$f_e(p)dp = \delta(p - p_0) \tag{110}$$

So, the redistribution function of the CMB photons scattered once by such an electron population is given by:

$$P_1(s) = \int_0^{\infty} dp\delta(p - p_0)P_s(s; p) \tag{111}$$

Thus,

$$P_1(s) = P_s(s; p_0) \tag{112}$$

Once, $P_1(s)$ is known, it is possible to to evaluate the probability that a frequency change s is produced by n scattering, and hence $j_n(x)$ and finally $g(x)$.

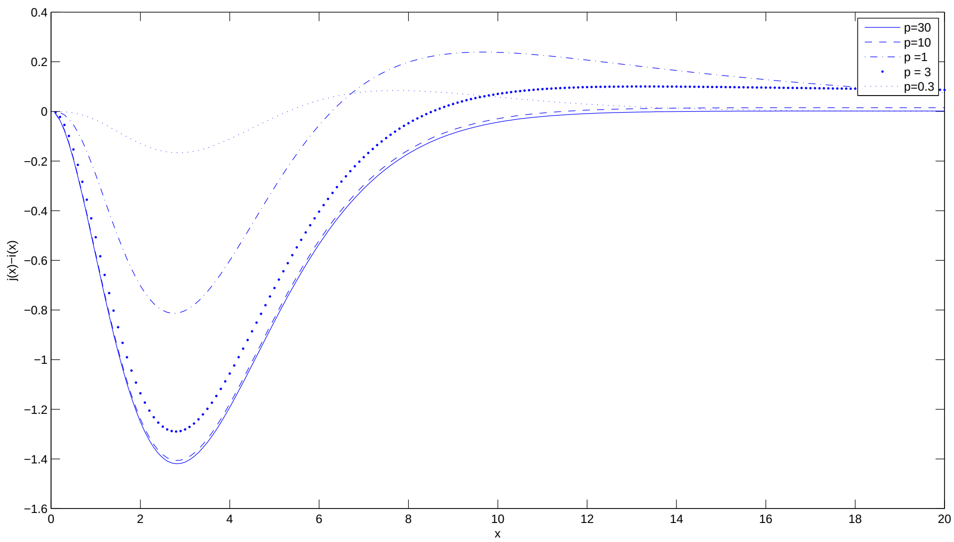

Figure 4: j(x)-i(x) for mono-energetic electron spectra with different p's

The curves given below give the spectral change per electron of a particular momentum. It is noted that the absorption feature saturates to $-i(x)$ for $p > 10$.

2.7 Conclusion: Comparison of distortion by thermal and non-thermal population of electron having same energy density

In this section, we've compared the distortion produced by thermal and non-thermal population(single power law, with $\alpha = 2.5$) of electrons in a spherical volume. Let us take, $E = 10^{60} ergs$, and the radius of the spherical volume enclosing this electron population $R = 100Kpc$. Following is the plots comparing there corresponding spectral shape g(x).

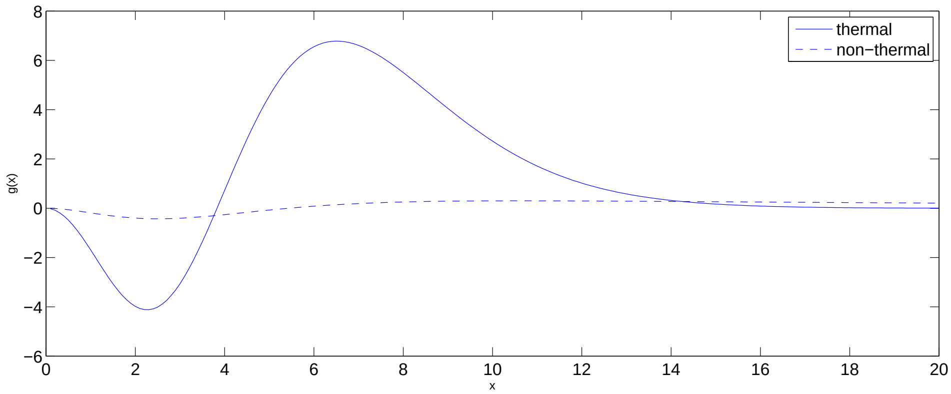

Figure 5: Comparison of spectral shape $g(x)$ for thermal and non-thermal population of electron.

$$P = \frac{E}{V}(\gamma - 1)$$

$$P = nk_BT_e \rightarrow for \quad thermal$$

$$P = n < k_B\tilde{T}_e > \rightarrow for \quad non-thermal$$

$$y_{th/non-th} = \frac{\sigma_T}{m_e c^2} \int P_{th/non-th} \; dl$$

Taking $\sigma_T = 6.65 \times 10^{-29} m^2$, $3 \times 10^{22} m$, and $\gamma_{th} = \frac{5}{3}$; $\gamma_{non-th} = \frac{4}{3}$, we get,

$$y_{th} = \frac{2}{3}(\frac{E}{V})\frac{\sigma_T}{m_e c^2}$$
$$y_{non-th} = \frac{1}{3}(\frac{E}{V})\frac{\sigma_T}{m_e c^2}$$
$$\therefore \Delta I_{th} = y_{th} g(x)_{th}$$
$$\Delta I_{non-th} = y_{non-th} g(x)_{non-th}$$

Also we plotted, ΔI Vs x, and as can be seen from the plot that the amplitude of ΔI in case of thermal population is more than that of non-thermal one. This can be explained from the fact that for thermal case(i.e; Maxwellian) the distribution of electrons is limited to a small range of energy. While for non thermal distribution the electrons are distributed over large energy range. So, for same energy density, the distortion in the CMB spectrum will be more from the thermal population as compared to non-thermal one.

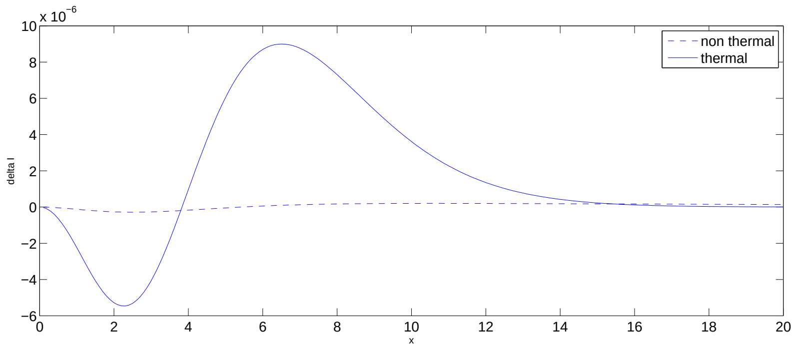

Figure 6: Comparison of ΔI for thermal and non-thermal population of electron.

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